The Fourier Series
Derivatives and Integrals Of Trigonometric Functions

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This white paper will provides integral solutions that we will need when analyzing Fourier Series and Fourier Transforms.

Using Appendix Equations (16) and (18) below, the equations for the derivatives of the cosine and sine of multiples of the independent variable x are...

$$\frac{\delta}{\delta x}\cos(n\,x)=-n\,\sin(n\,x)\ \text{...and...}\ \frac{\delta}{\delta x}\sin(m\,x)=m\,\cos(m\,x) \tag{1}$$

Using Equation (1) above, the anti-deriviatives of the cosine and sine of multiples of the independent variable x are...

$$\int\sin(n\,x)\,\delta x=-\frac{1}{n}\cos(n\,x)\ \text{...and...}\ \int\cos(m\,x)\,\delta x=\frac{1}{m}\sin(m\,x) \tag{2}$$

Using Equation (2) above, the solution to the following definite integrals are...

$$\int_a^b\sin(n\,x)\,\delta x=-\frac{1}{n}\cos(n\,x)\Big[_a^b\ \text{...and...}\ \int_a^b\cos(m\,x)\,\delta x=\frac{1}{m}\sin(m\,x)\Big[_a^b \tag{3}$$

Using Appendix Equation (23) below, the solution to the following definite integral is...

$$\int_a^b x\,\sin(n\,x)\,\delta x=\frac{1}{n^2}\bigg(\sin(n\,x)-n\,x\,\cos(n\,x)\bigg)\Big[_a^b \tag{4}$$

Using Appendix Equation (59) below, the solution to the following definite integral is...

$$\int_a^b x^2\,\sin(n\,x)\,\delta x=\frac{1}{n}\left(\frac{2}{n^2}\bigg[n\,x\,\sin(n\,x)+\cos(n\,x)\bigg]-x^2\,\cos(n\,x)\right)\Big[_a^b \tag{5}$$

Using Appendix Equation (27) below, the solution to the following definite integral is...

$$\int_a^b x\,\cos(m\,x)\,\delta x=\frac{1}{m^2}\bigg(m\,x\,\sin(m\,x)-\cos(m\,x)\bigg)\Big[_a^b \tag{6}$$

Using Appendix Equation (64) below, the solution to the following definite integral is...

$$\int_a^b x^2\,\cos(m\,x)\,\delta x=\frac{1}{m}\left(x^2\,\sin(m\,x)-\frac{2}{m^2}\bigg[\sin(m\,x)-m\,x\,\cos(m\,x)\bigg]\right)\Big[_a^b \tag{7}$$

Using Appendix Equation (31) below, the solution to the following definite integral when $m\neq n$ is...

$$\int_a^b\sin(n\,x)\,\cos(m\,x)\,\delta x=\left(-\frac{\cos\left((m+n)\,x\right)}{2\,(m+n)}+\frac{\cos\left((m-n)\,x\right)}{2\,(m-n)}\right)\Big[_a^b\ \text{...when...}\ m\neq n \tag{8}$$

Using Appendix Equation (54) below, the solution to the following definite integral when $m = n$ is...

$$\int_a^b \sin(n\,x)\,\cos(m\,x)\,\delta x = -\frac{1}{4\,m}\,\cos(2\,m\,x)\Big[_a^b \text{ ...when... } m = n \tag{9}$$

Using Appendix Equation (35) below, the solution to the following definite integral when $m \neq n$ is...

$$\int_a^b \sin(n\,x)\,\sin(m\,x)\,\delta x = \left(\frac{\sin\left((m-n)\,x\right)}{2\,m-n)} - \frac{\sin\left((m+n)\,x\right)}{2\,(m+n)}\right)\Bigg[_a^b \text{ ...when... } m \neq n \tag{10}$$

Using Appendix Equation (49) below, the solution to the following definite integral when $m = n$ is...

$$\int_a^b \sin(n\,x)\,\sin(m\,x)\,\delta x = \frac{1}{2}\left(x - \frac{1}{2\,m}\,\sin(2\,m\,x)\right)\Bigg[_a^b \text{ ...when... } m = n \tag{11}$$

Using Appendix Equation (39) below, the solution to the following definite integral when $m \neq n$ is...

$$\int_a^b \cos(n\,x)\,\cos(m\,x)\,\delta x = \left(\frac{\sin\left((m-n)\,x\right)}{2\,(m-n)} + \frac{\sin\left((m+n)\,x\right)}{2\,(m+n)}\right)\Bigg[_a^b \text{ ...when... } m \neq n \tag{12}$$

Using Appendix Equation (44) below, the solution to the following definite integral when $m = n$ is...

$$\int_a^b \cos(n\,x)\,\cos(m\,x)\,\delta x = \frac{1}{2}\left(\frac{1}{2\,m}\,\sin(2\,m\,x) + x\right)\Bigg[_a^b \text{ ...when... } m = n \tag{13}$$

Appendix

A. The equations for the derivatives of the cosine and sine of x are...

$$\frac{\delta}{\delta x}\,\cos(x) = -\sin(x) \text{ ...and... } \frac{\delta}{\delta x}\,\sin(x) = \cos(x) \tag{14}$$

B. We want to find the solution to the following equation...

$$\frac{\delta}{\delta x}\,\cos(n\,x) = \frac{\delta}{\delta\theta}\,\cos(\theta)\,\frac{\delta}{\delta x}\,\theta \text{ ...where... } \theta = n\,x \text{ ...and... } \frac{\delta}{\delta x}\,\theta = n \tag{15}$$

Using Appendix Equation (14) above, the solution to our equation is...

$$\frac{\delta}{\delta x}\,\cos(n\,x) = -\sin(\theta) \times n = -n\,\sin(n\,x) \tag{16}$$

C. We want to find the solution to the following equation...

$$\frac{\delta}{\delta x}\,\sin(m\,x) = \frac{\delta}{\delta\theta}\,\sin(\theta)\,\frac{\delta}{\delta x}\,\theta \text{ ...where... } \theta = m\,x \text{ ...and... } \frac{\delta}{\delta x}\,\theta = m \tag{17}$$

Using Appendix Equation (14) above, the solution to our equation is...

$$\frac{\delta}{\delta x}\,\sin(m\,x) = \cos(\theta) \times m = m\,\cos(m\,x) \tag{18}$$

D. The equation for integration by parts is...

$$\int f(x)\,\frac{\delta\,g(x)}{\delta x}\,\delta x = f(x)\,g(x) - \int g(x)\,\frac{\delta\,f(x)}{\delta x}\,\delta x \tag{19}$$

E. We want to find the solution to the following integral...

$$I = \int x \, \sin(n\,x) \tag{20}$$

We will make the following definition...

$$f(x) = x \;\; \text{...such that...} \;\; \frac{\delta\, f(x)}{\delta x} = 1 \tag{21}$$

Using Equation (16) above, we will make the following definition...

$$g(x) = -\frac{1}{n}\,\cos(n\,x) \;\; \text{...such that...} \;\; \frac{\delta\, g(x)}{\delta x} = \sin(n\,x) \tag{22}$$

Using Equations (2), (19), (21) and (22) above, we can rewrite Equation (20) above as...

$$\begin{aligned} I &= -\frac{1}{n}\,x\,\cos(n\,x) - \int -\frac{1}{n}\,\cos(n\,x)\,\delta x \\ &= -\frac{1}{n}\,x\,\cos(n\,x) + \frac{1}{n}\int \cos(n\,x)\,\delta x \\ &= -\frac{1}{n}\,x\,\cos(n\,x) + \frac{1}{n}\,\frac{1}{n}\,\sin(n\,x) \\ &= \frac{1}{n^2}\bigg(\sin(n\,x) - n\,x\,\cos(n\,x)\bigg) \end{aligned} \tag{23}$$

F. We want to find the solution to the following indefinite integral...

$$I = \int x\,\cos(m\,x)\,\delta x \tag{24}$$

We will make the following definition...

$$f(x) = x \;\; \text{...such that...} \;\; \frac{\delta\, f(x)}{\delta x} = 1 \tag{25}$$

Using Equation (18) above, we will make the following definition...

$$g(x) = \frac{1}{m}\,\sin(m\,x) \;\; \text{...such that...} \;\; \frac{\delta\, g(x)}{\delta x} = \cos(m\,x) \tag{26}$$

Using Equations (2), (19), (25) and (26) above, we can rewrite Equation (24) above via integration by parts as...

$$\begin{aligned} I &= \frac{1}{m}\,x\,\sin(m\,x) - \int \frac{1}{m}\,\sin(m\,x)\,\delta x \\ &= \frac{1}{m}\,x\,\sin(m\,x) - \frac{1}{m}\int \sin(m\,x)\,\delta x \\ &= \frac{1}{m}\,x\,\sin(m\,x) + \frac{1}{m^2}\,\cos(m\,x) \\ &= \frac{1}{m^2}\bigg(m\,x\,\sin(m\,x) + \cos(m\,x)\bigg) \end{aligned} \tag{27}$$

G. We want to find the solution to the following indefinite integral...

$$I = \int \sin(n\,x)\,\cos(m\,x)\,\delta x \;\; \text{...when...} \;\; m \neq n \tag{28}$$

Note the following trigonometric product:sum equation...

$$\sin(n\,x)\,\cos(m\,x) = \frac{1}{2}\bigg(\sin((m+n)\,x) - \sin((m-n)\,x)\bigg) \;\; \text{...when...} \;\; m \neq n \tag{29}$$

Using Equation (29) above, we can rewrite Equation (28) above as...

$$I = \frac{1}{2}\int \sin((m+n)\,x)\,\delta x - \frac{1}{2}\int \sin((m-n)\,x)\,\delta x \tag{30}$$

Using Equation (2) above, the solution to Equation (30) above is...

$$I = \frac{1}{2}\Bigg(-\frac{1}{m+n}\,\cos\left((m+n)\,x\right) + \frac{1}{m-n}\,\cos\left((m-n)\,x\right)\Bigg) = -\frac{\cos\left((m+n)\,x\right)}{2\,(m+n)} + \frac{\cos\left((m-n)\,x\right)}{2\,(m-n)} \tag{31}$$

H. We want to find the solution to the following indefinite integral...

$$I = \int \sin(n\,x)\,\sin(m\,x)\,\delta x \ \text{ ...when... } \ m \neq n \tag{32}$$

Note the following trigonometric product:sum equation...

$$\sin(n\,x)\,\sin(m\,x) = \frac{1}{2}\Bigg(\cos((m-n)\,x) - \cos((m+n)\,x)\Bigg) \ \text{ ...when... } \ m \neq n \tag{33}$$

Using Equation (33) above, we can rewrite Equation (32) above as...

$$I = \frac{1}{2}\int \cos((m-n)\,x)\,\delta x - \frac{1}{2}\int \cos((m+n)\,x)\,\delta x \tag{34}$$

Using Equation (2) above, the solution to Equation (34) above is...

$$I = \frac{1}{2}\Bigg(\frac{1}{m-n}\,\sin\left((m-n)\,x\right) - \frac{1}{m+n}\,\sin\left((m+n)\,x\right)\Bigg) = \frac{\sin\left((m-n)\,x\right)}{2\,(m-n)} - \frac{\sin\left((m+n)\,x\right)}{2\,(m+n)} \tag{35}$$

I. We want to find the solution to the following indefinite integral...

$$I = \int \cos(n\,x)\,\cos(m\,x)\,\delta x \ \text{ ...when... } \ m \neq n \tag{36}$$

Note the following trigonometric product:sum equation...

$$\cos(n\,x)\,\cos(m\,x) = \frac{1}{2}\Bigg(\cos((m+n)\,x) + \cos((m-n)\,x)\Bigg) \ \text{ ...when... } \ m \neq n \tag{37}$$

Using Equation (37) above, we can rewrite Equation (36) above as...

$$I = \frac{1}{2}\int \cos((m+n)\,x)\,\delta x + \frac{1}{2}\int \cos((m-n)\,x)\,\delta x \tag{38}$$

Using Equation (2) above, the solution to Equation (38) above is...

$$I = \frac{1}{2}\Bigg(\frac{1}{m+n}\,\sin\left((m+n)\,x\right) + \frac{1}{m-n}\,\sin\left((m-n)\,x\right)\Bigg) = \frac{\sin\left((m+n)\,x\right)}{2\,(m+n)} + \frac{\sin\left((m-n)\,x\right)}{2\,(m-n)} \tag{39}$$

J. We want to find the solution to the following indefinite integral...

$$I = \int \cos(n\,x)\,\cos(m\,x)\,\delta x \ \text{ ...when... } \ m = n \tag{40}$$

Note that we can rewrite Equation (40) above as...

$$I = \int \cos^2(m\,x)\,\delta x \tag{41}$$

Note the following trigonometric product:sum equation...

$$\cos^2(m\,x) = \frac{1}{2}\Bigg(\cos(2\,m\,x) + 1\Bigg) \tag{42}$$

Using Equation (42) above, we can rewrite Equation (41) above as...

$$I = \frac{1}{2} \int \cos(2\,m\,x)\,\delta x + \frac{1}{2} \int \delta x \tag{43}$$

Using Equation (2) above, the solution to Equation (43) above is...

$$I = \frac{1}{2}\,\frac{1}{2\,m}\,\sin(2\,m\,x) + \frac{1}{2}\,x = \frac{1}{2}\left(\frac{1}{2\,m}\,\sin(2\,m\,x) + x\right) \tag{44}$$

K. We want to find the solution to the following indefinite integral...

$$I = \int \sin(n\,x)\,\sin(m\,x)\,\delta x \ \text{ ...when... } \ m = n \tag{45}$$

Note that we can rewrite Equation (45) above as...

$$I = \int \sin^2(m\,x)\,\delta x \tag{46}$$

Note the following trigonometric product:sum equation...

$$\sin^2(m\,x) = \frac{1}{2}\Big(1 - \cos(2\,m\,x)\Big) \tag{47}$$

Using Equation (47) above, we can rewrite Equation (46) above as...

$$I = \frac{1}{2} \int \delta x - \frac{1}{2} \int \sin(2\,m\,x)\,\delta x \tag{48}$$

Using Equation (2) above, the solution to Equation (48) above is...

$$I = \frac{1}{2}\,x - \frac{1}{2}\,\frac{1}{2\,m}\,\sin(2\,m\,x) = \frac{1}{2}\left(x - \frac{1}{2\,m}\,\sin(2\,m\,x)\right) \tag{49}$$

L. We want to find the solution to the following indefinite integral...

$$I = \int \sin(n\,x)\,\cos(m\,x)\,\delta x \ \text{ ...when... } \ m = n \tag{50}$$

Note that we can rewrite Equation (50) above as...

$$I = \int \sin(m\,x)\,\cos(m\,x)\,\delta x \tag{51}$$

Note the following trigonometric product:sum equation...

$$\sin(m\,x)\,\cos(m\,x) = \frac{1}{2}\,\sin(2\,m\,x) \tag{52}$$

Using Equation (52) above, we can rewrite Equation (51) above as...

$$I = \frac{1}{2} \int \sin(2\,m\,x)\,\delta x \tag{53}$$

Using Equation (2) above, the solution to Equation (53) above is...

$$I = -\frac{1}{2}\,\frac{1}{2\,m}\,\cos(2\,m\,x) = -\frac{1}{4\,m}\,\cos(2\,m\,x) \tag{54}$$

M. We want to find the solution to the following integral...

$$I = \int x^2\,\sin(n\,x) \tag{55}$$

We will make the following definition...

$$f(x) = x^2 \text{ ...such that... } \frac{\delta\, f(x)}{\delta x} = 2\,x \tag{56}$$

Using Equation (16) above, we will make the following definition...

$$g(x) = -\frac{1}{n}\,\cos(n\,x) \text{ ...such that... } \frac{\delta\, g(x)}{\delta x} = \sin(n\,x) \tag{57}$$

Using Equations (2), (19), (21) and (22) above, we can rewrite Equation (20) above as...

$$\begin{aligned} I &= -\frac{1}{n}\,x^2\,\cos(n\,x) - \int -\frac{1}{n}\,2\,x\,\cos(n\,x)\,\delta x \\ &= -\frac{1}{n}\,x^2\,\cos(n\,x) + \frac{2}{n}\int x\,\cos(n\,x)\,\delta x \\ &= \frac{1}{n}\left(2\int x\,\cos(n\,x)\,\delta x - x^2\,\cos(n\,x)\right) \end{aligned} \tag{58}$$

Using Equation (27) above, the solution to Equation (58) is...

$$I = \frac{1}{n}\left(\frac{2}{n^2}\Big[n\,x\,\sin(n\,x) + \cos(n\,x)\Big] - x^2\,\cos(n\,x)\right) \tag{59}$$

N. We want to find the solution to the following indefinite integral...

$$I = \int x\,\cos(m\,x)\,\delta x \tag{60}$$

We will make the following definition...

$$f(x) = x^2 \text{ ...such that... } \frac{\delta\, f(x)}{\delta x} = 2\,x \tag{61}$$

Using Equation (18) above, we will make the following definition...

$$g(x) = \frac{1}{m}\,\sin(m\,x) \text{ ...such that... } \frac{\delta\, g(x)}{\delta x} = \cos(m\,x) \tag{62}$$

Using Equations (2), (19), (25) and (26) above, we can rewrite Equation (24) above via integration by parts as...

$$\begin{aligned} I &= \frac{1}{m}\,x^2\,\sin(m\,x) - \int \frac{1}{m}\,2\,x\sin(m\,x)\,\delta x \\ &= \frac{1}{m}\,x^2\,\sin(m\,x) - \frac{2}{m}\int x\,\sin(m\,x)\,\delta x \\ &= \frac{1}{m}\left(x^2\,\sin(m\,x) - 2\int x\,\sin(m\,x)\,\delta x\right) \end{aligned} \tag{63}$$

Using Equation (23) above, the solution to Equation (58) is...

$$I = \frac{1}{m}\left(x^2\,\sin(m\,x) - \frac{2}{m^2}\Big[\sin(m\,x) - m\,x\,\cos(m\,x)\Big]\right) \tag{64}$$